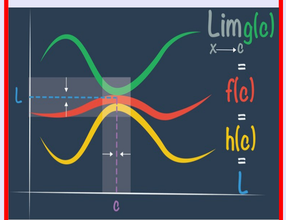
# Math 261

## Spring 2022

## Lecture 18




Consider $f(x) = (x+2)^3$

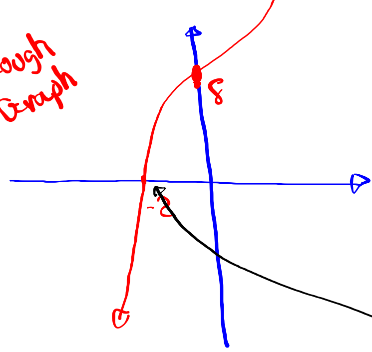
1) Domain $f(x)$ is a polynomial function
   Domain $(-\infty, \infty)$

2) y-Int $\rightarrow f(0) = (0+2)^3 = 8 \Rightarrow (0,8)$

3) x-Int. $\rightarrow f(x)=0 \rightarrow (x+2)^3=0 \Rightarrow x=-2 \Rightarrow (-2,0)$

4) $f(x)$ behaves like $y=x^3$, shift left 2 units.

Rough Graph




5) Critical number
   $f'(x)=0$ or undefined
   $f'(x) = 3(x+2)^2 \cdot 1$
   $f'(x) = 3(x+2)^2$
   $f'(x)=0 \rightarrow x=-2$
   C.P. $\rightarrow (-2,0)$

6) Possible Inflection Point

$f''(x) = 0$ or undefined

$f''(x) = 3 \cdot 2(x+2)^{1} \cdot 1$

$f''(x) = 6(x+2)$

$f''(x) = 0 \Rightarrow x = -2 \Rightarrow (-2, 0)$

$f'(x) = 3(x+2)^2$

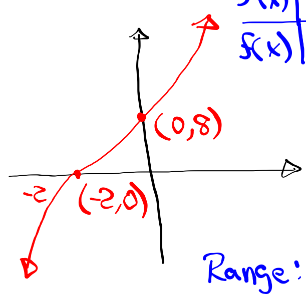
7) Sign chart

| $x$ | $-\infty$ | $-2$ | $\infty$ |
|---|---|---|---|
| $f'(x)$ | + | • | + |
| $f''(x)$ | − | • | + |
| $f(x)$ | ↗ |  | ↗ |

Inc., C.D.    Inc., C.U.

No local Max

NO local Min




Range: $(-\infty, \infty)$

Consider $f(x) = x^{1/3}(x+4)$    $x^{1/3} = \sqrt[3]{x}$

odd root
No restrictions

1) Domain $(-\infty, \infty)$

2) Y-Int $\Rightarrow f(0) = 0^{1/3}(0+4) = 0$

$\Rightarrow (0, 0)$

3) x-Int $\Rightarrow f(x) = 0 \Rightarrow x^{1/3}(x+4) = 0$

$x = 0 \quad x = -4$

$\Rightarrow (0,0), (-4,0)$

4) C.N. $\Rightarrow f'(x) = 0$ OR $f'(x)$ undefined

$f(x) = x^{1/3}(x+4)$

$f'(x) = \frac{1}{3}x^{-2/3}(x+4) + x^{1/3} \cdot 1$

$= \frac{1}{3}x^{-2/3}(x+4) + \frac{3}{3}x^{1/3} = \frac{1}{3}x^{-2/3}\left[x+4+3x^{1}\right]$

$= \frac{1}{3}x^{-2/3}(4x+4)$

$= \frac{4}{3}x^{-2/3}(x+1)$

$= \frac{4(x+1)}{3x^{2/3}}$

$f'(x) = 0 \rightarrow x+1 = 0 \rightarrow x = -1$

$f'(x)$ undefined $\rightarrow x^{2/3} = 0 \rightarrow x = 0$

C.P. $(-1, f(-1)) = (-1, -3)$

$(0, f(0)) = (0, 0)$

find P.I.P.

$f''(x)=0$ or $f''(x)$ undefined

$-\frac{2}{3}-1=\frac{-2}{3}-\frac{3}{3}$
$=-\frac{5}{3}$

$\frac{-5}{3}+1=$
$\frac{-5}{3}+\frac{3}{3}=$
$\frac{-2}{3}$

$$f'(x)=\frac{4}{3}x^{-2/3}(x+1)$$

$$f''(x)=\frac{4}{3}\left[\frac{-2}{3}x^{-5/3}(x+1)+x^{-2/3}\cdot 1\right]$$

$$=\frac{4}{3}\left[\frac{-2}{3}x^{-5/3}(x+1)+\frac{3}{3}x^{-2/3}\right]$$

$$=\frac{4}{3}\cdot\frac{1}{3}\cdot x^{-5/3}\left[-2(x+1)+3x^{1}\right]$$

$$=\frac{4}{9}x^{-5/3}\left[-2x-2+3x\right]=\frac{4(x-2)}{9x^{5/3}}$$

$f''(x)=0 \to x-2=0 \to x=2$

$f''(x)$ is undefined $\to x^{5/3}=0 \to x=0$

P.I.P. $(2,f(2))=(2,6\sqrt[3]{2})$

$(0,f(0))=(0,0)$

Sign Chart

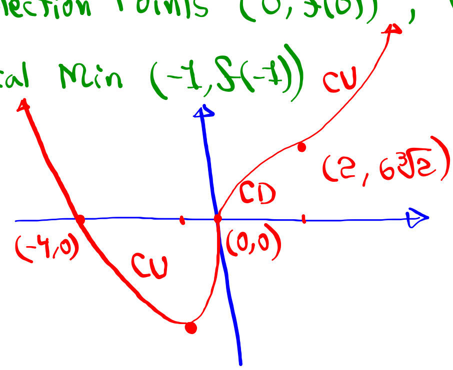
$f'(x)=\frac{4(x+1)}{3x^{2/3}}$ $f''(x)=\frac{4(x-2)}{9x^{5/3}}$

| $x$ | $-\infty$ | $-1$ | | $0$ | | $2$ | $\infty$ |
|---|---|---|---|---|---|---|---|
| $f'(x)$ | $-$ | ● | $+$ | ○ | $+$ | | $+$ |
| $f''(x)$ | $+$ | | $+$ | ○ | $-$ | ● | $+$ |
| $f(x)$ | ↘ | | ↗ | | ↗ | | ↗ |

Inflection Points $(0,f(0))$, $(2,f(2))$

Local Min $(-1,f(-1))$

$f(-1)=\sqrt[3]{-1}\,(-1+4)$
$=-3$



April 18, 2022

Prove a third-degree polynomial function has exactly one inflection point.

$f(x) = ax^3 + bx^2 + cx + d \,,\; a \neq 0$

$f'(x) = 3ax^2 + 2bx + c$

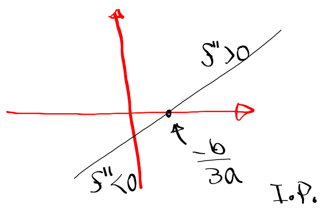
$f''(x) = 6ax + 2b$

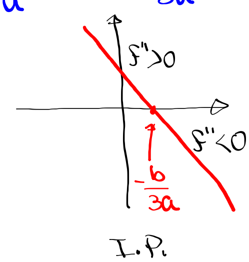
$f''(x)$ is a polynomial $\Rightarrow$ Defined everywhere

$f''(x) = 0 \Rightarrow 6ax + 2b = 0$

$x = \dfrac{-2b}{6a}$ $\quad x = \dfrac{-b}{3a}$

Linear eqn




$f(x) = \dfrac{x}{x+2}$

1) Domain $\quad (-\infty,-2) \cup (-2,\infty)$

$x+2 \neq 0$

$x \neq -2$

2) All intercepts $\quad$ Y-Int $(0,0)$

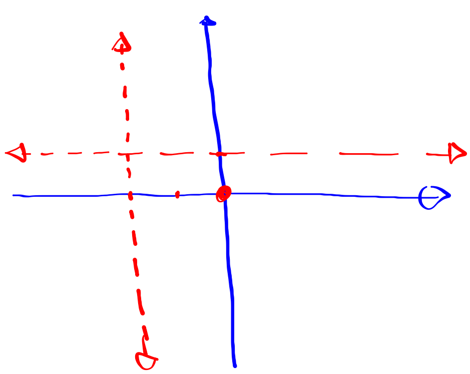
x-Int $(0,0)$

3) All Asymptotes

V.A. $\Rightarrow x = -2$

H.A. $\Rightarrow y = \lim_{x\to\infty} f(x) = 1$

$y = \lim_{x\to -\infty} f(x) = 1$



4) $f'(x) = \frac{1(x+2) - x\cdot 1}{(x+2)^2} = \frac{2}{(x+2)^2} > 0$ f(x) is increasing

$= 2(x+2)^{-2}$

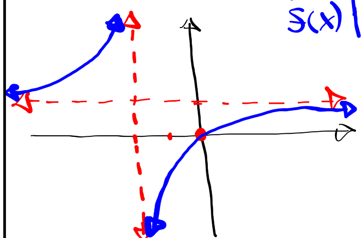
5) $f''(x) = 2\cdot(-2)\cdot(x+2)^{-3}\cdot 1$

$= \frac{-4}{(x+2)^3}$ P.I.P. at $x = -2$

Not in the domain

6) Sign chart

| $x$ | $-\infty$ | $-2$ | $\infty$ |
|---|---|---|---|
| $f'(x)$ | $+$ | $0$ | $+$ |
| $f''(x)$ | $+$ | $0$ | $-$ |
| $f(x)$ | ↗ | | ↗ |



Consider $f(x) = \frac{x^2 - 1}{x^3}$

1) Domain $(-\infty, 0) \cup (0, \infty)$

$x^3 \neq 0$, $x \neq 0$ Y-Int $\rightarrow$ None

2) All intercepts $x$-Int $\rightarrow (1,0), (-1,0)$

3) All Asymptotes V.A. $\rightarrow x = 0$

H.A. $\rightarrow y = \lim_{x\to\infty} f(x) = 0$

$y = \lim_{x\to-\infty} f(x) = 0$

4) Discuss Symmetry

$f(-x) = \frac{(-x)^2 - 1}{(-x)^3} = \frac{x^2 - 1}{-x^3} = -\frac{x^2 - 1}{x^3} = -f(x)$

$f(-x) = -f(x) \Rightarrow f(x)$ is an odd function

Symmetric with respect to the origin.

$f'(x) = \dfrac{3-x^2}{x^4}$ , $f''(x) = \dfrac{2(x^2-6)}{x^5}$ $f(x) = \dfrac{x^2-1}{x^3}$

5) C.N. & C.P.

$f'(x)=0$ or undefined

$3-x^2=0$ $x^4=0$

$x=\pm\sqrt{3}$

$\left(\sqrt{3}, \frac{2}{3\sqrt{3}}\right)$

$\left(-\sqrt{3}, \frac{-2}{3\sqrt{3}}\right)$

6) P.I.P.

$f''(x)=0$ or undefined

$x^2-6=0$ $x^4=0$

$x=\pm\sqrt{6}$

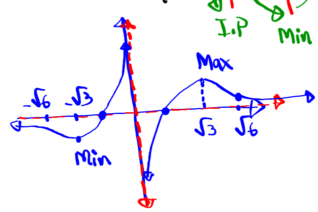
$\left(\sqrt{6}, \frac{5}{6\sqrt{6}}\right)$

$\left(-\sqrt{6}, \frac{-5}{6\sqrt{6}}\right)$

7) Sign Chart

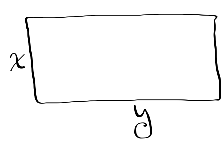
| $x$ | $-\infty$ to $-\sqrt{6}$ | $-\sqrt{6}$ to $-\sqrt{3}$ | $-\sqrt{3}$ to $0$ | $0$ to $\sqrt{3}$ | $\sqrt{3}$ to $\sqrt{6}$ | $\sqrt{6}$ to $\infty$ |
|---|---|---|---|---|---|---|
| $f'(x)$ | − | − | + | + | − | − |
| $f''(x)$ | − | + | + | − | − | + |
| $f(x)$ | ↘ I.P | ↘ Min | ↗ | ↗ Max | ↘ I.P. | ↘ |

8) Graph




Perimeter of a rectangular garden is 100 ft.

Find its dimensions with maximum area.




$P = 100$

$2x + 2y = 100 \rightarrow x+y=50$

$y = 50-x$

Area $= xy$

$= x(50-x)$

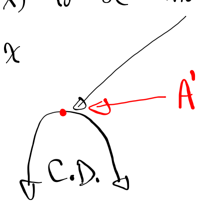
$A(x) = 50x - x^2$

we need $A(x)$ to be maximum

$A'(x) = 50 - 2x$

$A''(x) = -2$

$A''(x) < 0$



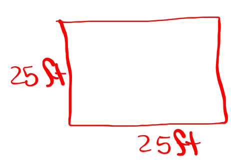

$A'(x)=0$

$50-2x=0$

$\boxed{x=25}$

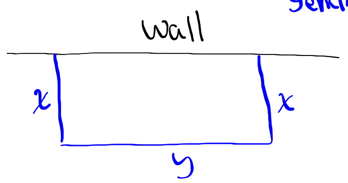
$y=50-x$

$\boxed{y=25}$




Area $= 25^2 = 625$ ft$^2$

We have 1000 ft of fencing.
We need a rectangular area by the side of barn for the dogs. No fence needed next to the wall.
Find the dimensions with max. Area.




Fencing $y + 2x = 1000$

$\text{Area} = xy$

$A(x) = x(1000 - 2x)$

$A(x) = 1000x - 2x^2$

$A'(x) = 1000 - 4x$

$A''(x) = -4$ C.D.

$A'(x) = 0$ ← Max

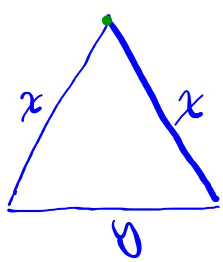
$1000 - 4x = 0$

$\boxed{x = 250}$

$\boxed{y = 500}$

Dimensions are 250 ft by 500 ft.

An isosceles triangle has a perimeter 12 m.
Find all sides that gives max. Area




$P = 12$

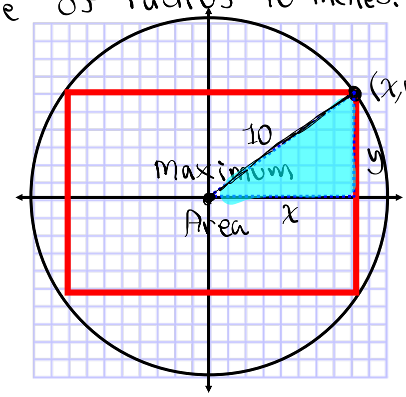
$\boxed{y + 2x = 12}$

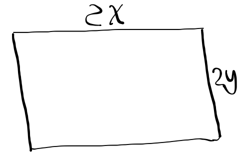
$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$$

Heron's Formula

**Review Heron's formula for Wednesday**

Find dimensions of a rectangle with maximum area that can be inscribed in a Circle of radius 10 inches.




Maximize

$Area = 4xy$

$x^2 + y^2 = 10^2$

$y^2 = 100 - x^2$

$y = \sqrt{100 - x^2}$

$A(x) = 4x\sqrt{100 - x^2}$

$A'(x) =$

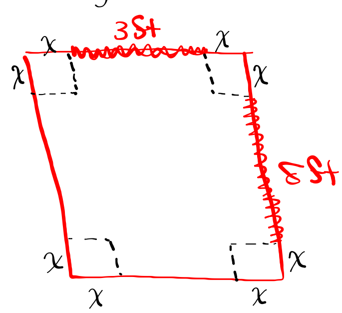
$A''(x) =$

Make sure to find $A'(x)$ & $A''(x)$ by Wed.

A sheet of metal has a rectangular shape, and it is 3 ft by 8 ft.

Cut 4 equal size Squares from 4 Corners.

Now bend up sides to make an open-top box.

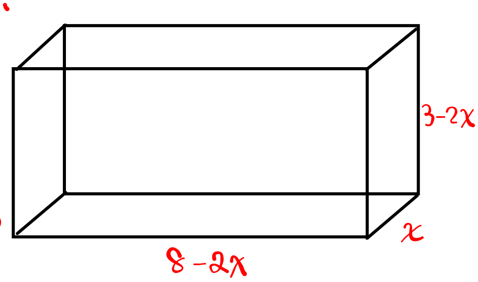



Volume of this Box:

$V = LWH$

$V = (8-2x)(3-2x) \cdot x$

Find x that makes Max. Volume.



$V = (8-2x)(3-2x) \cdot x$

$= (24 - 16x - 6x + 4x^2)x$

$= (4x^2 - 22x + 24)x$

$V(x) = 4x^3 - 22x^2 + 24x$

$V'(x) =$

$V''(x) =$

Solve $V'(x) = 0$, determine for which one

Max where $V'(x) = 0$

$V''(x) < 0$